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Stability of a finite-difference discretization of a singular perturbation problem

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ABSTRACT

A new higher-order finite-difference scheme is proposed for a linear singularly perturbed convection–diffusion problem in one dimension. It is shown how the theory of inverse-monotone matrices, the Lorenz decomposition in particular, can be applied to the stability analysis of the resulting linear system.

1. Introduction

At the ALA 2010 conference, I presented a talk on some matrix-theory results as applied to the proof of stability for various finite-difference discretizations of different types of singularly perturbed boundary-value problems in one dimension. Since the readers of this journal have limited interest in singular perturbations, this paper is an abbreviated version of the talk. Instead of considering several discretization schemes, I focus here on one newly-constructed higher-order scheme for a singularly perturbed convection–diffusion problem. This still illustrates how some matrix-related results can be used in the stability analysis of finite-difference schemes for singular perturbation problems.

For the purpose of this paper, it suffices to consider the problems of the following simplified form:

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Find a $C^2[0, 1]$-solution u of

$$-\varepsilon u'' - b(x)u' + c(x)u = 0, \quad x \in [0, 1], \quad u(0) = U_0, \quad u(1) = U_1, \tag{1}$$

where ε is the perturbation parameter ($0 < \varepsilon \le \varepsilon^* \ll 1$), b and c are sufficiently smooth functions, and U_0 and U_1 are given constants. It should be remarked that all results presented here can be generalized to the semilinear case $c = c(x, u)$.

General discussions of numerical methods for singular perturbation problems can be found in Miller et al. [13], Roos et al. [16], Farrell et al. [4], and Linß [10]. When singular perturbation problems are solved numerically, it is important to obtain errors which converge to 0 uniformly in ε as the discretization parameter N (the number of mesh subintervals) tends to ∞. This is known as convergence uniform in ε and, ideally, it should be present at every point of the discretization mesh. Classical numerical methods for boundary value problems, generally speaking, do not produce numerical solutions that converge uniformly in ε and this is why special numerical methods have to be constructed for singular perturbation problems. The main difficulty is in the fact that solutions to singularly perturbed problems typically have one or more boundary and/or interior layers, i.e., narrow intervals in which the derivatives are unbounded as $\varepsilon \to 0$.

Convergence uniform in ε can be obtained by either using exponentially fitted schemes, like in [6], or special discretization meshes which are dense in the layer(s), like in most of the other works cited here. Two types of *a priori* constructed meshes are well-known. Bakhvalov meshes [1] are formed by a mesh generating function which appropriately redistributes equidistantly spaced points, giving a mesh dense in the layer(s). The more recent Shishkin meshes [17] are simpler because they are piecewise equidistant, fine in the layer(s) and coarse outside the layer(s). It is not surprising that the smoother Bakhvalov meshes give better theoretical and numerical results than the Shishkin ones (see the comparisons in [15,11,24], for instance). On Bakhvalov meshes the errors behave like $\mathcal{O}(N^{-r})$, where the positive number r is the rate of convergence. On the other hand, the convergence on Shishkin meshes is slowed down by logarithmic factors since the errors typically behave like $\mathcal{O}(N^{-r} \ln^s N)$ for some positive constants r and s. This is referred to as "convergence of order *almost* r".

Nevertheless, Shishkin meshes are more suitable for higher-order finite-difference schemes which use more than three mesh points. These schemes are easier to construct and analyze on an equidistant mesh, so they can be applied on equidistant parts of the Shishkin mesh. At the same time, some simpler non-equidistant schemes can be used at a few points of the transition region between the fine and coarse parts of the mesh. One such hybrid scheme is proposed here for problem (1) in the case when function b is of constant sign. The scheme is constructed with an accuracy of almost third order in mind, which is an improvement over the existing second-order schemes [18,7–9]. However, the present interest is only to prove that the scheme is stable and its higher-order uniform accuracy is illustrated by numerical results. A proof of uniform convergence for the proposed scheme would by no means be simple and it anyway would not be of interest to the readership of this journal.

When a singular perturbation problem like (1) is discretized, a system of linear equations, $Aw^N = 0$, is obtained, where w^N is the vector representing the numerical solution. Stability uniform in ε means that the matrix A is nonsingular and that, in some suitable matrix norm,

$$\|A^{-1}\| \le M \tag{2}$$

with a positive constant M independent of ε and N. This desirable property of all discretizations of singular perturbation problems can be proved in different ways, depending on the problem and the discretization scheme. In the simplest cases, the proof is based either on strict diagonal dominance or on M-matrices (inverse-monotone L-matrices). As schemes become more complicated for the purpose of increased accuracy, more sophisticated methods of proof have to be applied, like different appropriate decompositions of the matrix. In this paper, the Lorenz standard decomposition [12] is used to prove (2).

Further notation, terminology, and other preliminaries are introduced in the next section. After this, the new scheme is described and its stability analyzed in Section 3. Finally, numerical results are presented in Section 4.

2. Preliminaries

Throughout the paper, generic positive constant independent of both ε and the discretization parameter N are denoted by M if they are sufficiently large and by m if they are sufficiently small. Some particular constants of this kind will be indexed.

Let $X = [0, 1]$ and let X^N be a discretization mesh with points x_i, $i = 0, 1, \dots, N$, $0 = x_0 < x_1 < \cdots < x_N = 1$, and steps $h_i = x_i - x_{i-1}$, $i = 1, 2, \dots, N$. Let also $\hbar_i = (h_i + h_{i+1})/2$ and let for any number $s \in (0, 1)$, $x_{i+s} = x_i + sh_{i+1}$ and $x_{i-s} = x_i - sh_i$. Mesh functions defined on $X^N \backslash \{0, 1\}$ are denoted by $w^N = (w_i^N)$, $v^N = (v_i^N)$, etc. We formally set $w_0^N = U_0$ and $w_N^N = U_1$ for any mesh function w^N. Each mesh function is identified with the corresponding column-vector, thus $w^N = [w_1^N, w_2^N, \dots, w_{N-1}^N]^T$. In particular, $e^N = [1, 1, \dots, 1]^T$. If g is a continuous function on X, we write g_i for $g(x_i)$. By $\|\cdot\|$ we denote the maximum vector norm, $\|w^N\| = \max_{1\le i\le N-1} |w_i^N|$, as well as its subordinate matrix norm.

For an $(N-1) \times (N-1)$-matrix A, let $A_d = \operatorname{diag}\{a_{11}, a_{22}, \dots, a_{N-1,N-1}\}$ and $A_o = A - A_d$. Let also $A^- = [a_{ij}^-]$ with

$$a_{ij}^- = \begin{cases} a_{ij} & \text{if } a_{ij} \le 0, \\ 0 & \text{otherwise.} \end{cases}$$

Matrix A is an L-matrix if $A_d > 0$ and $A_o \le 0$ (inequalities involving vectors and matrices should be understood component-wise). An inverse-monotone matrix is a nonsingular matrix A for which $A^{-1} \ge 0$. An inverse-monotone L-matrix is called an M-matrix.

For singularly perturbed reaction–diffusion problems, for which $b \equiv 0$ and $c \ge c_* > 0$ on X, discretization schemes typically give strictly diagonally dominant (SDD) matrices [21,5,19,23]. If A is SDD, the estimate (2) can be proved using the following Varah principle.

Principle 1 [20]**.** Assume A is strictly diagonally dominant by rows and set $\alpha = \min_i \left(|a_{ii}| - \sum_{j\ne i} |a_{ij}|\right)$, $\alpha > 0$. Then $\|A^{-1}\| < 1/\alpha$.

Another approach that is often used to prove (2) for discretizations of singular perturbation problems is based on M-matrices:

Principle 2 [2,12,3]**.** Let A be an L-matrix and let there exist a vector v^N such that $v^N > 0$ and $Av^N \ge \beta e^N$ for some positive constant β. A is then an M-matrix and it holds that $\|A^{-1}\| \le \beta^{-1} \|v^N\|$.

In the absence of an L-matrix, the following result can be used:

Principle 3 [12]**.** For a matrix $A = [a_{ij}]$, let A_o^- be decomposed so that $A_o^- = A^z + A^s$, where $A^z = [a_{ij}^z] \le 0$ and $A^s = [a_{ij}^s] \le 0$, and that the following conditions are satisfied:

(i) $A_d + A^z$ is an M-matrix (hence $a_{ii} > 0$, $i = 1, 2, \dots, N-1$);
(ii) for every $a_{ij} > 0$, $i \ne j$, it holds that

$$a_{ij} \le \sum_{k=1}^{N-1} a_{ik}^z a_{kk}^{-1} a_{kj}^s;$$

(iii) there exist a vector v^N such that $v^N > 0$ and $Av^N \ge \beta e^N$ for some positive constant β.

A is then a product of two M-matrices (and thus inverse monotone) and it holds that $\|A^{-1}\| \le \beta^{-1}\|v^N\|$.

The particular convection–diffusion problem (1) considered here is assumed to satisfy

$$b(x) \geq b_* > 0 \quad \text{and} \quad c^* \geq c(x) \geq 0 \quad \text{for } x \in X. \tag{3}$$

Numerical methods for this kind of problem are discussed in [4,6–11,13–16,18,22,24]. Under Conditions (3), problem (1) has a unique solution, the derivatives of which can be estimated as follows (see [6]):

$$\left|u^{(k)}(x)\right| \leq M\left(1+\varepsilon^{-k}e^{-b_*x/\varepsilon}\right), \quad x \in X,\ k = 0, 1, \ldots .$$

Since these estimates are sharp, this shows that, in general, u has an exponential boundary layer of width $\mathcal{O}(\varepsilon|\ln\varepsilon|)$ near $x = 0$.

The Shishkin mesh for problem (1), denoted by S^N, is adjusted to this behavior of the solution. It consists of two equidistant parts. The intervals $[0, \tau]$ and $[\tau - 1]$ are divided, respectively, into J and $N - J$ equidistant subintervals, where $\tau = a\varepsilon \ln N$, a is a positive parameter, and J is a positive integer such that $Q = J/N$ is kept fixed and $Q < 1$, $1/Q \leq M$. It is assumed that $\tau < 1$ since N is unrealistically large otherwise. The first part of the mesh is used inside the boundary layer. It has J fine mesh steps $h = \tau/J$, whereas the other part has $N - J$ coarse mesh steps $H = (1-\tau)/(N-J)$. The point $x_J = \tau$ is the transition point between the fine and coarse parts of the mesh.

3. The discretization

In this section, the problem (1), satisfying (3), is discretized by a higher-order scheme on the mesh S^N.

We define the following equidistant four-point finite-difference operators, which are the same as in [22]. Let χ stand for the step size of an equidistant mesh. $D^{(k)}_{\chi,s}$ approximates $u^{(k)}(x_{i+s})$ for $k = 2, 1, 0$:

$$D^{(2)}_{\chi,s}w^N_i = \frac{1}{\chi^2}\left[(1-s)w^N_{i-1} + (3s-2)w^N_i + (1-3s)w^N_{i+1} + sw^N_{i+2}\right],$$

$$D^{(1)}_{\chi,s}w^N_i = \frac{1}{6\chi}\left[(-3s^2+6s-2)w^N_{i-1} + 3(3s^2-4s-1)w^N_i\right.$$
$$\left. +3(-3s^2+2s+2)w^N_{i+1} + (3s^2-1)w^N_{i+2}\right],$$

$$D^{(0)}_{\chi,s}w^N_i = \frac{1}{2}\left[s(s-1)w^N_{i-1} + 2(1-s^2)w^N_i + s(s+1)w^N_{i+1}\right].$$

The accuracy of $D^{(2)}_{\chi,s}$ is in general of second order only, but it increases to third order if $s = \sigma := (3-\sqrt{15})/6 \approx -.145$. When $s = 0$, we get the standard equidistant central scheme for $u''(x_i)$. $D^{(1)}_{\chi,s}$ and $D^{(0)}_{\chi,s}$ are both third-order accurate for any value of s. The value $s = \theta := 1/\sqrt{3}$ is of particular interest since it makes $D^{(1)}_{\chi,s}$ a three-point operator. It is interesting to compare $D^{(1)}_{\chi,s}$ to the so-called κ-scheme, see [27, p. 149] and the references therein. The κ-scheme is a one-parameter family of schemes intended to discretize u' with higher accuracy. It reaches third order only as a four-point scheme, thus $D^{(1)}_{\chi,\theta}$ is simpler. The following scheme is an equidistant discretization of the continuous operator in (1) at the point x_{i+s}:

$$\Lambda_{\chi,s}w^N_i := -\varepsilon D^{(2)}_{\chi,s}w^N_i - b_{i+s}D^{(1)}_{\chi,s}w^N_i + c_{i+s}D^{(0)}_{\chi,s}w^N_i.$$

We apply this discretization at all points of the mesh S^N where this is possible to do, using $s = \sigma$ and $\chi = h$, or $s = \theta$ and $\chi = H$. At the points where the equidistant four-point schemes cannot be applied, the midpoint upwind scheme [18] is used, defined by

$$\Lambda w_i^N := -\varepsilon D'' w_i^N - b_{i+1/2} D' w_i^N + c_{i+1/2} \frac{w_i^N + w_{i+1}^N}{2} = 0,$$
$$D'' w_i^N = \frac{1}{\hbar_i}\left(\frac{w_{i+1}^N - w_i^N}{h_{i+1}} - \frac{w_i^N - w_{i-1}^N}{h_i}\right),$$
$$D' w_i^N = \frac{w_{i+1}^N - w_i^N}{h_{i+1}}.$$

In the scheme Λ, $-b(x)u' + c(x)u$ is approximated at $x_{i+1/2}$ with second order accuracy.

Thus, we construct the following discrete problem corresponding to (1):

$$Lw_i^N = 0, \ \ i = 1, 2, \ldots, N-1, \tag{4}$$

where

$$Lw_i^N = \begin{cases} \Lambda_{h,\sigma} w_i^N & \text{for } 1 \le i \le J-2, \\ \Lambda w_i^N & \text{for } i = J-1, J, \\ \Lambda_{H,\theta} w_i^N & \text{for } J+1 \le i \le N-2, \\ \Lambda w_i^N & \text{for } i = N-1. \end{cases}$$

We prove next that the discrete operator L is stable, i.e., that the corresponding matrix A satisfies (2). This requires the assumption

$$\varepsilon^* \le \frac{M_*}{N}, \tag{5}$$

where M_* is a suitable constant which can be determined and which is independent of both ε and N.

Theorem 1. *Let (3) and (5) hold true. Then the discretization of problem (1) which uses scheme (4) on the mesh S^N is stable for all values of $\varepsilon \in (0, \varepsilon^*]$ provided N is sufficiently large independently of ε.*

Proof. The elements of matrix A satisfy

$$a_{ii} > 0, \quad a_{i,i\pm 1} \le 0, \quad i = 1, 2, \ldots, N-1$$

and

$$a_{i,i+2} \begin{cases} > 0 & \text{for } i = 1, 2, \ldots, J-2, \\ \le 0 & \text{for } i = J-1, J, \ldots, N-3. \end{cases}$$

The remaining entries of matrix A are equal to 0. All of the above inequalities are straightforward to prove except for the following ones:

$$a_{i,i-1} \le 0, \quad i = 2, 3, \ldots, J-2 \tag{6}$$

and

$$a_{i,i+1} \le 0, \quad i = J+1, J+2, \ldots, N-2. \tag{7}$$

With i like in (6), we have

$$\begin{aligned} a_{i,i-1} &= -\frac{3+\sqrt{15}}{6h^2}\varepsilon + \frac{2+\sqrt{15}}{12h} b_{i+\sigma} + \frac{1}{12} c_{i+\sigma} \\ &\le -m_1 \frac{N^2}{\varepsilon \ln^2 N} + M_1 \frac{N}{\varepsilon \ln N} + \frac{c^*}{12} = \frac{N}{\varepsilon \ln N}\left(M_1 - m_1 \frac{N}{\ln N}\right) + \frac{c^*}{12} \end{aligned}$$

from where we can see that (6) holds true if N is sufficiently large but independent of ε.

Let now i be like in (7). We have

$$a_{i,i+1} = \frac{3\theta - 1}{H^2}\varepsilon + \frac{1-2\theta}{2H}b_{i+\theta} + \frac{1+3\theta}{6}c_{i+\theta}$$
$$\leq M_2N^2\varepsilon - m_2N + M_3 \leq (M_2M_* - m_2)N + M_3,$$

where we have used (5) in the last step. Therefore, if M_* is chosen so that $M_* < m_2/M_2$, (7) follows provided N is sufficiently large independently of ε.

Since some elements $a_{i,i+2}$ are positive, A is not an L-matrix. It is not SDD either, thus Principles 1 and 2 cannot be applied. We therefore proceed with Principle 3. Its Condition (iii) is easy to satisfy. Since $L(2 - x_i) \geq b_*$, $i = 1, 2, \ldots, N-1$, we choose v^N with $v_i^N = 2 - x_i$ to get $Av^N \geq b_*e^N$. We have $\|A^{-1}\| \leq b_*^{-1}\|v^N\| \leq 2/b_*$, and then (2) holds true, provided we prove Conditions (i) and (ii).

To this end, we construct the Lorenz standard decomposition [12] of A_o^- (which is equal to A^-). We consider the following blocks within matrix A:

$$\begin{bmatrix} a_{k-1,k} & a_{k-1,k+1} \\ a_{kk} & a_{k,k+1} \end{bmatrix}, \tag{8}$$

where $a_{k-1,k+1} > 0$. For an element $a_{i,i+1} < 0$, the following cases are of interest:

(a) $a_{i,i+1}$ appears in exactly one block of type (8) and the positive off-diagonal element within this block is in the ith row (this is the case only with a_{12});
(b) $a_{i,i+1}$ appears in exactly two blocks of type (8).

Then the elements of matrix A^z are formed according to

$$a_{ij}^z = \begin{cases} a_{ij} & \text{if case (a) holds true, i.e., if } i = j-1 = 1, \\ \frac{1}{2}a_{ij} & \text{if case (b) holds true, i.e., if } i = j-1 = 2, 3, \ldots, J-2, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $P := A_d + A^z$ is a matrix with elements p_{ij} being other than 0 only for $i = j = 1, 2, \ldots, N-1$ and $i = j-1 = 1, 2, \ldots, J-2$. We prove next that $Pe^N \geq me^N$. Condition (i) is then satisfied because of Principle 2. It holds true that

$$p_{ii} + p_{i,i+1} \geq \begin{cases} a_{ii} + a_{i,i+1} & \text{if } i = 1, 2, \ldots, J-2, \\ a_{ii} & \text{if } i = J-1, J, \ldots, N-1. \end{cases}$$

For $i = 1, 2, \ldots, J-2$, we have

$$a_{ii} + a_{i,i+1} \geq \frac{1}{2h}\left[\frac{2\varepsilon}{h} - b_{i+\sigma}(1-2\sigma)\right] \geq m$$

because $h = (a/Q)\varepsilon N^{-1}\ln N$. For the same reason, $a_{J-1,J-1} \geq m$. In all other cases, i.e., for $i = J, J+1, \ldots, N-1$,

$$a_{ii} \geq m_3b_*N^{-1} \geq m.$$

Let us finally prove Condition (ii). Consider $a_{i,i+2} > 0$, $i = 1, 2, \ldots, J-2$. Condition (ii) is satisfied if we show that

$$a_{i,i+2} \leq a_{i,i+1}^z a_{i+1,i+1}^{-1} a_{i+1,i+2}^s. \tag{9}$$

By construction, both $a^z_{i,i+1}$ and $a^s_{i+1,i+2}$ in (9) are negative and

$$a^z_{i,i+1}a^s_{i+1,i+2} \geq \frac{1}{4}a_{i,i+1}a_{i+1,i+2}.$$

Therefore, we need to prove

$$4a_{i+1,i+1}a_{i,i+2} \leq a_{i,i+1}a_{i+1,i+2}. \tag{10}$$

In each element a_{ij} in (10), only the term denoted by a''_{ij}, which results from the scheme for u'', should be considered, since it dominates over any other term (if present). This is because ε/h^2 dominates over M/h and any constant term. However, the terms a''_{ij} satisfy the corresponding strict inequality,

$$4a''_{i-1,i-1}a''_{i,i-2} < a''_{i,i-1}a''_{i-1,i-2}. \tag{11}$$

Indeed, for $i = 1, 2, \ldots, J-3$, we just verify that $4(-s)(2-3s) < (3s-1)^2$, and for $i = J-2$, $4(-s)\cdot 2 < (3s-1)(-1)$. $\square$

The Lorenz standard decomposition has been used previously in the stability analysis of finite-difference schemes for singular perturbation problems. In paper [14], the same problem is considered like here, but the scheme discussed is equidistant and of a different kind. The above technique of proof is more similar to [25,26], where the discretization is done on a Shishkin mesh. However, different problem types are considered there, viz. a two-parameter problem and a reaction–diffusion problem, respectively. In these two papers, the Lorenz standard decomposition is combined with an additional decomposition of the matrix.

Strictly speaking, the result of Theorem 1 does not mean ε-uniform stability because of the Condition (5). However, this condition is not a big practical constraint since it usually holds true that $\varepsilon \ll N^{-1}$.

4. Numerical results

In this section, we present some numerical results which justify the construction and use of the hybrid scheme L. We use the following problem for our numerical experiments:

$$-\varepsilon u'' - (x+1)u' + u = f(x), \quad x \in X, \quad u(0) = u(1) = 0,$$

where the function f is chosen so that

$$u(x) = e^{-x/\varepsilon} - e^x + (e - e^{-1/\varepsilon})x.$$

Tables 1 and 2 below show the maximum pointwise errors

$$E(N) = \left\| w^N - u^N \right\|,$$

where w^N is the numerical solution and $u^N_i = u(x_i)$. The numerical rate of accuracy is calculated as usual by

$$R(N) = \log_2[E(N)/E(2N)].$$

The Shishkin mesh S^N is used with different values of the parameters a and Q. The presented results are for $\varepsilon = 10^{-6}$. We have also tested $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-9}$ and the results are very similar. This means that the method shows uniformity with respect to ε.

It is to be expected that the second-order accuracy of $D^{(2)}_{H,\theta}$ does not affect the accuracy of the scheme L because $D^{(2)}_{H,\theta}$ is used sufficiently far from the layer and is multiplied by ε which satisfies (5). The less accurate scheme Λ, used at three mesh points, should not spoil overall accuracy either. Tables 1 and 2 show that all errors are relatively small but that the rate of convergence is below 3. However,

Table 1
Results on Shishkin mesh for $a = 4$ and $\varepsilon = 10^{-6}$.

N	250		500		1000
Q	$E(N)$	$R(N)$	$E(N)$	$R(N)$	$E(N)$
0.5	9.93–5	2.43	1.84–5	2.51	3.24–6
0.7	3.76–5	2.45	6.86–6	2.53	1.19–6
0.9	1.23–5	2.28	1.53–6	2.41	4.72–7

Table 2
Results on Shishkin mesh for $a = 3$ and $\varepsilon = 10^{-6}$.

N	250		500		1000
Q	$E(N)$	$R(N)$	$E(N)$	$R(N)$	$E(N)$
0.5	4.28–5	2.44	7.87–6	2.51	1.38–6
0.7	1.60–5	2.46	2.90–6	2.53	5.03–7
0.9	9.27–6	2.97	1.18–6	2.97	1.51–7

the latter can be attributed to the Shishkin mesh because of the expected slow-down of convergence by logarithmic factors.

Tables 1 and 2 illustrate how the results change when the parameters a and Q are changed. The results are better when a is less and Q is greater, since the mesh is then denser in the layer. However, a cannot be decreased much further – we report here that the errors for $a = 2$ are considerably worse. This too is what can be expected of the Shishkin mesh, on which the proof of ε-uniform convergence requires that ab_* be bounded from below by a positive constant independent of N and ε.

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